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# Coarse-Graining Approach to First-Order Phase Transitions

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On an example of a simple spin system with two ground states and no symmetry, we show how to control low-temperature systems near first-order phase transitions by a straightforward renormalization group argument. The method, as opposed to the Pirogov–Sinai approach, also works for complex Hamiltonians.

**KEY WORDS:** First-order phase transitions; renormalization group; contour models.

## 1. INTRODUCTION

The renormalization group[(12)] has been devised to study phenomena with no underlying short length scale, such as second-order phase transitions and the behavior near them, or the related short-distance properties of the quantum field theory. In both problems the scale is set by the correlation length, which is large in dimensionless units (the units of ultraviolet cutoff). The renormalization group replaces such a problem by a sequence of problems with short-range scales introduced by hand.

As opposed to the case of the second-order phase transitions, around the first-order transition the correlation length stays finite and short for low temperature and there seems to be no *a priori* reason for the multiscale analysis. There is, however, a second length scale present in the problem: the size of the critical droplet in the false vacuum.[4] To understand this, within an equilibrium setup, let us look at the low-temperature Ising model

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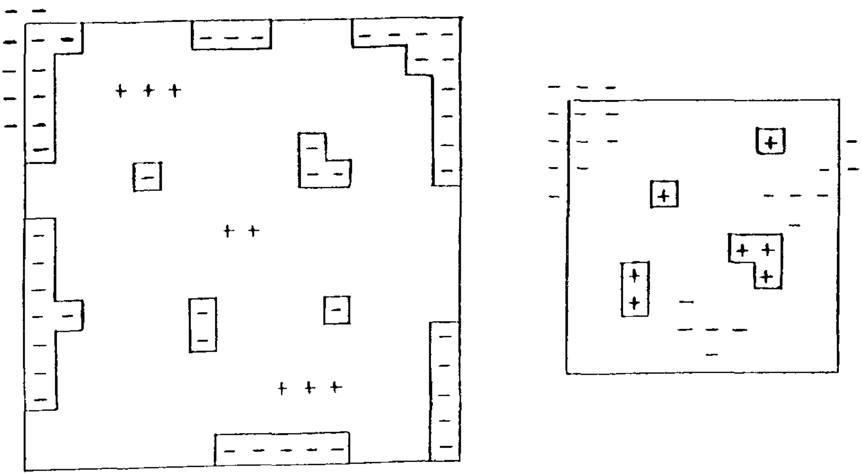

Figure 1

in dimension $d \geqslant 2$ with a small, positive magnetic field. Consider the $(-)$-boundary condition for a finite box $\Lambda$. If the box is big, then the typical configuration will have a big contour near $\partial\Lambda$, which flips the boundary conditions, and a few small contours inside the $(+)$-sea surrounded by it. If the box is smaller, then the typical configuration will just have a few small contours in the negative sea (see Fig. 1).

The size $L$ of the borderline box is roughly given by equating the energy of the additional contour near $\partial\Lambda$ with the gain of the energy by flipping the inside of $\Lambda$ from the minus to plus sign:

$$2d\beta L^{d-1} \approx 2hL^d$$

Thus,

$$L = O(\beta/h)$$

so that it diverges at the transition point. $L$ sets a scale independent of the correlation length $\xi$, which is $O(\beta^{-1})$ and stays finite at $h = 0$.

It is natural to use a multiscale analysis to deal with a diverging scale. The heuristic renormalization-group treatment of the first-order phase transitions may be found in Refs. 7 and 2. It put stress on the presence of operators scaling like volume at the transition point. Our rigorous approach, not totally unrelated in spirit, proceeds as follows. If both $\xi$ and $L$ are small, we shall deal with the system by a single-scale low-temperature expansion. If $\xi$ is small but $L$ is not, however, we shall perform a coarse-graining transformation, which lowers $\xi$ and $L$ and iterate it until $L$ gets small enough for the single-scale expansion to converge.

The choice of the coarse-graining procedure in low temperature is somewhat subtle, since we want to be able to iterate the procedure easily and to avoid the pathologies of the type described in Ref. 3. This is done by working directly with the lowest lying excitations of the system: the Peierls contours. We reformulate our model as a "contour model with external field." The coarse graining consists in a resummation of the smallest contours, which is controlled in the thermodynamic limit by a convergent expansion, and of a blocking of the remaining ones. The application of the procedure leaves us in the class of contour models with external field, but modifies the value of the external field by addition of the (tiny) free energy density of small contours and by multiplication of the resulting value by the volume rescaling factor (the volume of the block). Thus, the external field expands under the coarse graining, providing a relevant variable in the renormalization group language.[7,2] This growth beats the growth of the effective (inverse) temperature $\beta$, so that $L$ decreases. The initial value of the field at the transition point where $L = \infty$ will be identified using the standard trick for the control of relevant variables due to Bleher and Sinai.[1] At this point, the coarse graining will be iterated an infinite number of times, driving the system to the low-temperature fixed point,[6] and the inductive multiscale expansion will establish the coexistence of different phases.

As compared with the existing rigorous theory of the first-order phase transitions due to Pirogov and Sinai,[9,11] our approach covers an important case beyond the scope of that theory: the case of complex Hamiltonians. Here the original version of the Pirogov–Sinai approach worked only at the transition point, but not in its neighborhood.[8] Statistical mechanical systems with complex Hamiltonians appear naturally in the study of metastability or in lattice gauge theories in the presence of the so-called $\theta$ topological terms, so that it is important to have an approach that makes it possible to treat first-order phase transitions in both cases.

This paper is organized as follows. In Section 2, starting with a simple Ising-type system in a magnetic field, we introduce the formalism of contour models with an external field. In Section 3, we treat such a contour model by a single-scale expansion in the case of low temperature and high external field (when both length scales discussed above are short). Section 4 is devoted to the treatment of the gas of small contours, a preparatory step to Section 5, describing the coarse-graining procedure for the contour models. Finally, in Section 6 the control of the low-temperature, small-field model is achieved by iterative application of the coarse-graining procedure, and Section 7 establishes the finiteness of the correlation length throughout the transition region.

In the present paper, for simplicity, we have dealt with systems with only two ground states. It should be clear that the approach may be generalized to the case with many ground states, leading to the extension of the Gibbs phase rule to systems with complex Hamiltonians. We shall leave this to a future publication.

When this paper was essentially completed, we learnt that another contour model approach to systems with first-order transitions[13] may be extended to the case of complex Hamiltonians. This approach, although working with single-scale contours, contains ideas also used in our analysis.

## 2. FROM SPIN MODEL TO CONTOUR MODEL

Instead of considering the most general spin or field theory system to which our approach applies, let us start with a simple example of the Ising-type system on the hypercubic lattice $\mathbb{Z}^d$, $d \geqslant 2$, with the Hamiltonian

$$\beta H(\sigma) = \tfrac{1}{2}\beta \sum_{\substack{\{x,y\} \\ |x-y|=1}} (\sigma_x - \sigma_y)^2 - \sum_{\substack{X \\ \operatorname{diam} X < R}} J_X \sigma^X \tag{1}$$

where $\sigma^X \equiv \prod_{x \in X} \sigma_x$ and $J_X$ are translationally invariant. We take $\beta$ and $J_X$ complex with $J_X$ bounded for $X \neq \{x\}$ and $\operatorname{Re} \beta \equiv \beta_0$ large, so that our model will have two approximate ground states with the energies shifted with respect to each other by means of $\operatorname{Re} J_{\{x\}}$.

In order to get rid of the infinite sums in (1), consider a finite volume $\Lambda$ given by the $l^N \times \cdots \times l^N$ closed cube in $\mathbb{R}^d$ centered at the origin, where $l$ is an odd integer. Introduce the finite-volume Hamiltonian $H_\Lambda(\sigma)$ by restricting in (1) the sums to lattice subsets intersecting $\Lambda$. For $\{x_j\} \subset \Lambda$, define the finite-volume correlation functions with $(+)$- or $(-)$-boundary condition as

$$\left\langle \prod_{j=1}^{J} \sigma_{x_j} \right\rangle_\Lambda^{\pm} = \sum_{\substack{\sigma_x = \pm 1 \\ x \in \Lambda}} \prod_{j=1}^{J} \sigma_{x_j} e^{-\beta H_\Lambda(\sigma)} \Bigg/ \sum_{\substack{\sigma_x = \pm 1 \\ x \in \Lambda}} e^{-\beta H_\Lambda(\sigma)} \tag{2}$$

with values of $\sigma$ outside $\Lambda$ fixed to plus or minus, respectively. Of course, (2) makes sense only if the denominator does not vanish, which will be shown later in the regime of interest, where we shall also control the *thermodynamic limit* $\Lambda \nearrow \mathbb{R}^d$ of (2).

It will be convenient to rewrite the statistical sums of (2) in terms of sums over contours involving the lattice sites where the spin configuration changes values. Given $\{x_j\}$ and a configuration $\sigma$ equal to $+1$ $(-1)$ outside $\Lambda$, call a site $x \in \Lambda \cap \mathbb{Z}^d$ a contour site if (i) there exists $y \in \mathbb{Z}^d$, $|x-y| < R$, such that $\sigma_y \neq \sigma_x$, or (ii) $x$ is in a closed $L \times \cdots \times L$ block centered at a point of $L\mathbb{Z}^d$ (an "$L$-block") containing points $x_j$ or in an $L$-block intersecting the latter ones ($L = l^n > R$ will be chosen later in a $\beta$-dependent way and will be only loosely related to the scale discussed in the introduction).

A connected component $\gamma$ of the union of the closed sites will be called a *contour*. A boundary cell of a contour will be assigned the sign of the spin at the center of the exterior unit cube adjacent to it. Each component of $\partial\gamma$ obtains a sign in this way. We shall call $\gamma$ with the signs of components of $\partial\gamma$ specified a *signed contour* (with some abuse, also denoted by $\gamma$). A signed contour $\gamma$ will be called *positive* or *negative* according to the sign of the exteriormost component of $\partial\gamma$ (see Fig. 2). Note that whether a contour $\gamma$ with $\partial\gamma \cap \partial\Lambda \neq \varnothing$ is positive or negative is determined by the boundary condition. We shall denote by $V(\gamma)$ the set surrounded by the exteriormost component of $\partial\gamma$. The interior of $\gamma$, $\operatorname{Int}\gamma = V(\gamma)\backslash\gamma$, splits into $\operatorname{Int}^{\pm}\gamma$ according to the signs of the surrounding components of $\partial\gamma$ (Fig. 2).

Now, given a signed contour $\gamma$ and a set of sites $\{y_k\} \subset \gamma$, let

$$\rho_{\{y_k\}}(\gamma) = \sum_{\sigma} \prod_{k} \sigma_{y_k} \exp\left\{ -\tfrac{1}{2}\beta \sum_{\substack{\{x,y\} \\ x \text{ or } y \text{ in } \gamma}} (\sigma_x - \sigma_y)^2 + \sum_{u \in \gamma} \sum_{\substack{X \ni u \\ \operatorname{diam} X < R}} J_X \sigma^X / |X \cap \Lambda| \right\} \tag{3}$$

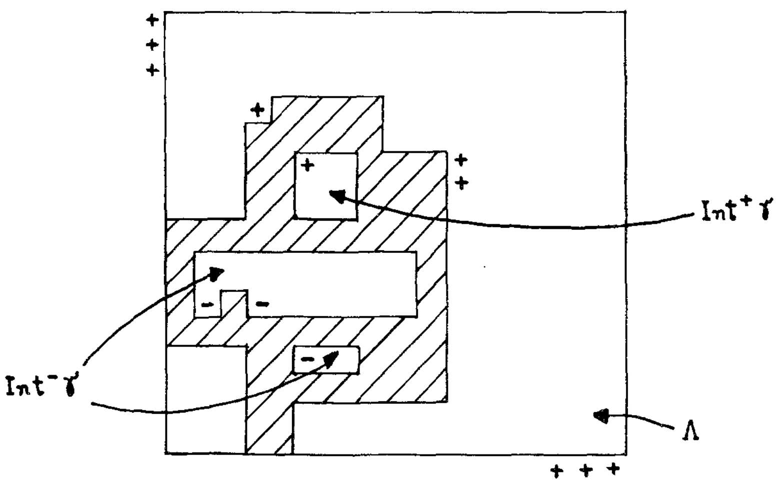


Figure 2

be the *activity* of $\gamma$. The first sum in (3) runs over all the configurations $\sigma$ for which $\gamma$ is the unique signed contour. We shall also denote

$$\rho_{\varnothing}(\gamma)=\rho(\gamma) \tag{4}$$

Notice that $\rho_{\{y_k\}}(\gamma)$ depends on $\Lambda$ only if $\mathrm{dist}(\gamma, \partial\Lambda)=0$.

Given a boundary condition, call a family $\partial=\{\gamma\}$ of signed contours in $\Lambda$ *compatible* if they are disconnected, if the signs on the components of their boundaries that can be connected outside $\bigcup\gamma$ agree, and if the signs of the exteriormost contours agree with the sign of the boundary condition. For a compatible family $\partial$, let $V^+(\partial)\cup V^-(\partial)=\Lambda\backslash\bigcup_{\gamma\in\partial}\gamma$ with the decomposition into $V^{\pm}(\partial)$ according to the signs on the boundary. Every configuration producing $\sigma$ is equal $+1$ $(-1)$ on $V^+(\partial)$ $[V^-(\partial)]$ (and on its $R$-neighborhood in $\Lambda$). Thus, each point $u\in V^{\pm}(\partial)$ contributes the energy

$$h_u^{\pm}=\sum_{\substack{X\ni u\\ \mathrm{diam}\, X<R}}(\pm 1)^{|X|}J_X/|X\cap\Lambda| \tag{5}$$

For $\mathrm{dist}(u, \partial\Lambda)\geqslant R$, $h_u^{\pm}=\mathrm{const}=h^{\pm}$.

The above definitions yield naturally the following expression for the correlation functions:

$$\begin{aligned}\left\langle\prod_j\sigma_{x_j}\right\rangle_\Lambda^{\pm}&=\left[\sum_{\substack{\text{compatible }\partial\\ \bigcup\gamma\supset\{x_j\},\,\gamma\in\partial}}\exp\left(\sum_{s=\pm}\sum_{u\in V^s(\partial)}h_u^s\right)\prod_{\gamma\in\partial}\rho_{\{x_j\}\cap\gamma}(\gamma)\right]\\&\quad\times\left[\sum_{\text{compatible }\partial}\exp\left(\sum_{s=\pm}\sum_{u\in V^s(\partial)}h_u^s\right)\prod_{\gamma\in\partial}\rho(\gamma)\right]^{-1}\\&\equiv Z_\Lambda^{\pm}(\{x_j\})/Z_\Lambda^{\pm}\end{aligned} \tag{6}$$

This is the contour model expression with which we shall work. It is a useful representation at low temperatures, where contours are strongly suppressed.

We shall need estimates on the contour activities expressing this suppression. For $\{y_k\}\subset\Lambda$ call the excluded volume $E(\{y_k\})$ the union of the $L$-blocks in $\Lambda$ containing points $y_k$ and of the $L$-blocks in $\Lambda$ intersecting the latter. Suppose that $|J_X|\leqslant O(1)$ for $|X|\geqslant 2$, $\mathrm{Re}\,\beta\equiv\beta_0\gg 1$, and that $\pm\mathrm{Re}(h^+-h^-)\equiv\pm 2\,\mathrm{Re}\,h\geqslant 0$. Then, as is easy to see,

$$\left|\rho_{\{y_k\}}(\gamma)\exp\left(-\sum_{u\in\gamma}h_u^{\pm}\right)\right|\leqslant A^{|\{y_k\}|}\exp[-2\eta\beta_0\,|\gamma\backslash E(\{y_k\})|] \tag{7}$$

for some $\eta>0$ and $A=O(1)$ ($|X|$ will denote the number of points for discrete sets $X\subset\mathbb{R}^d$ and the volume for nondiscrete ones).

## 3. LARGE EXTERNAL FIELD EXPANSION

Suppose that

$$\pm \operatorname{Re} h \equiv \pm h_0 \geqslant D \tag{8}$$

for $D$ big enough. To be definite, fix the upper sign in (8). Now not only the contours but also the $(-)$ islands are strongly suppressed and we may treat our system by a single-scale, low-temperature expansion. To this end, divide the set $(\bigcup_{\gamma \in \partial} \gamma) \cup V^-(\partial)$ into connected components (call them *polymers* $X_\alpha$). For $\{y_k\} \subset X_\alpha$ define the polymer activities

$$\rho^+_{\{y_k\}}(X_\alpha) = \exp\left(-\sum_{u \in X_\alpha} h_u^+\right) \sum_\partial \exp\left(\sum_{s=\pm}\ \sum_{u \in V^s(\partial) \cap X_\alpha} h_u^s\right) \times \prod_{\gamma \in \partial} \rho_{\{y_k\} \cap \gamma}(\gamma) \tag{9}$$

where $\partial$ runs through the compatible families of contours $\gamma$ in $X_\alpha$ such that $\partial X_\alpha$ coincides exactly with the union of positive components of $\partial \gamma$ [while $\partial(X \backslash \bigcup \gamma)$ consists of the negative ones] (see Fig. 3). We have

$$\left\langle \prod_j \sigma_{x_j} \right\rangle_A^+ = \sum_{\substack{\{X_\alpha\} \\ \text{disjoint} \\ \bigcup X_\alpha \supset \{x_j\}}} \prod_\alpha \rho^+_{\{x_j\} \cap X_\alpha}(X_\alpha) \Big/ \sum_{\substack{\{X_\alpha\} \\ \text{disjoint}}} \prod_\alpha \rho^+(X_\alpha) \tag{10}$$

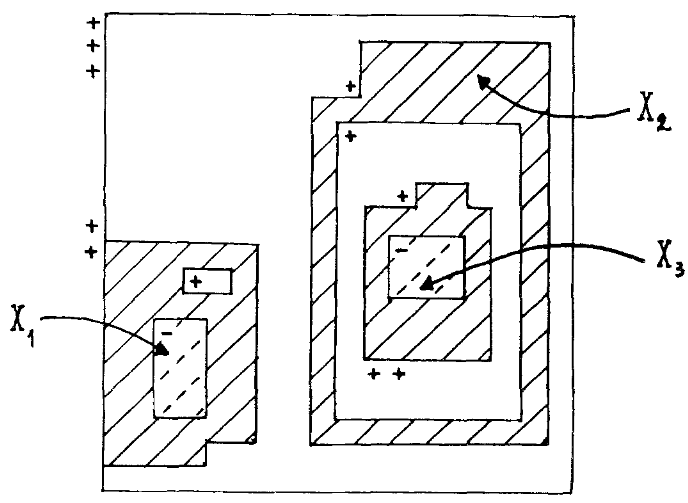


Figure 3

Notice that for $\beta_0$ and $D$ large enough,

$$|\rho^{\pm}_{\{y_k\}}(X_\alpha)| \leqslant A^{|\{y_k\}|} \exp[-\max(\eta\beta_0, D)\, |X_\alpha \backslash E(\{y_k\})|] \tag{11}$$

so that the polymer activities are strongly suppressed.

The control of (10) is standard (see, e.g., Refs. 10 and 5). We obtain

$$\left\langle \prod_j \sigma_{x_j} \right\rangle^+_\Lambda = \sum_{\substack{\{X_\alpha\} \text{ disjoint} \\ \bigcup X_\alpha \supset \{x_j\} \\ X_\alpha \cap \{x_j\} \neq \varnothing}} \prod_\alpha \rho^+_{\{x_j\} \cap X_\alpha}(X_\alpha)$$
$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(Y_1,\ldots,Y_n)} a\left(\left\{\bigcup X_\alpha, Y_m\right\}\right) \prod_{m=1}^{n} \rho^+(Y_m) \tag{12}$$

where, for a collection of sets $\{Z_\delta\}$,

$$a(\{Z_\delta\}) = \sum_G (-1)^{|G|} \tag{13}$$

with $G$ running through the connected graphs on the vertices $Z_\delta$ with lines between intersecting sets. Now, it is straightforward to establish the convergence of (12) uniform in the thermodynamic limit and the convergent expansion for $\Lambda = \mathbb{R}^d$, which is (12) with $X_\alpha$ and $Y_m$ running through the (bounded) polymers in $\mathbb{R}^d$ [notice that $\rho^+_{\{y_k\}}(X_\alpha)$ stabilize when $\Lambda \nearrow \mathbb{R}^d$]. This establishes the existence of the $(+)$ *phase* for $\beta_0$ and $D$ large enough and $h_0 \geqslant D$. In the same way for $-h_0 \geqslant D$, we establish the existence of the $(-)$ *phase.*

The argument that would show in the real case that the wrong $(\mp)$ boundary conditions lead to the same infinite-volume limit, since in a big, finite volume there would be a contour around $\partial\Lambda$ flipping the sign (see our discussion of the length scale $L$ in the introduction), does not extend to the complex case, since it requires lower bounds on sums of activities. What happens with the wrong boundary condition will not be decided here.

Instead, let us decide which is the right boundary condition for $|h_0| < D$. The argument proceeds by induction. The coarse graining, which resums small contours and blocks the big ones, produces an effective contour model on a longer scale with increased $\beta_0$ and expanded $h_0$. This process will be continued until $\pm h_0$ becomes bigger than $D$, so that we may apply the expansion of the present section, or until the effective volume becomes small. This way, we shall be able to pick, for the given imaginary part of $h$ (or of $J_{\{x\}}$), the real part $h_{0c}$ such that for $h_0 \gtrless h_{0c}$, the $\pm$ phase exists, and for $h_0 = h_{0c}$ both phases coexist.

Let us start with the treatment of small contours.

## 4. SMALL CONTOURS IN EXTERNAL FIELD

Let us assume that $|h_0| < 2D$ [for the control of the inductive procedure, it is convenient to consider here the whole interval $(-2D, 2D)$ of $h_0$, although for $|h_0| > D$ the existence of the corresponding phase was already settled above]. First, we shall rewrite the estimate (7) in a way more convenient for subsequent use:

$$\left|\rho_{\{y_k\}}(\gamma) \exp\left(-\sum_{u \in \gamma} h_u^{\pm}\right)\right| \leqslant A^{|\{y_k\}|} \exp[-\eta\beta_0 |\gamma \backslash E(\{y_k\})|] \qquad \text{for} \quad \gamma \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \tag{14}$$

for $\beta_0 > \bar{\beta}_0(\eta, D)$. The condition (14) holds uniformly in $\Lambda$. Moreover,

$$\left|\rho_{\{y_k\}}(\gamma) \exp\left(-\sum_{u \in \gamma} h_u^{\pm}\right) \restriction_{\Lambda} - \rho_{\{y_k\}}(\gamma) \exp\left(-\sum_{u \in \gamma} h_u^{\pm}\right) \restriction_{\mathbb{R}^d}\right|$$
$$\leqslant 2 \exp[-\varepsilon\beta_0 \operatorname{dist}(\gamma, \partial\Lambda)]\, A^{|\{y_k\}|} \exp[-\eta\beta_0 |\gamma \backslash E(\{y_k\})|]$$
$$\text{for} \quad \gamma \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \tag{15}$$

as $\rho_{\{y_k\}}(\gamma)$ depends on $\Lambda$ only if $\operatorname{dist}(\gamma, \partial\Lambda) = 0$.

Since pairs of bounds like (14) and (15) will play an important role in the sequel, below, by stating that a bound

$$|f(R)| \leqslant P \tag{16}$$

for $R \subset \Lambda$ holds uniformly in $\Lambda$, we shall also mean that

$$|f(R) \restriction_{\Lambda} - f(R) \restriction_{\mathbb{R}^d}| \leqslant 2\{\exp[-\varepsilon\beta_0 \operatorname{dist}(R, \partial\Lambda)]\}\, P \tag{17}$$

for some small $\varepsilon > 0$ (say, $\varepsilon = \eta/100$).

Let us divide the set of all (signed) contours into the sets of *small* and *big* ones. A contour will be called small if $|V(\gamma)| < L^d$, where $L = l^n \ll \beta_0/D$ will be chosen so that the suppression of small-contour activities overcomes the external field contributions from the interior of the contours; see below. Big contours are the ones with $V(\gamma) \geqslant L^d$. For any finite union of unit lattice cubes $V \subset \Lambda$ such that if a small contour $\gamma$ lies in $V$, then so does its interior, define

$$\tilde{Z}_V^{\pm} = \exp\left(-\sum_{u \in V} h_u^{\pm}\right) \sum_{\text{compatible } \tilde{\partial}} \exp\left(\sum_{s = \pm} \sum_{u \in V^s(\tilde{\partial})} h_u^s\right) \prod_{\gamma \in \tilde{\partial}} \rho(\gamma) \tag{18}$$

where $\tilde{\partial}$ runs through the collection of compatible small contours in $V$ and

$$V^{+}(\tilde{\partial}) \cup V^{-}(\tilde{\partial}) = V \Big\backslash \bigcup_{\gamma \in \tilde{\partial}} \gamma$$

Introducing

$$\tilde{\rho}(\gamma) = \rho(\gamma) \exp\left[ -\sum_{u \in \gamma} h_u^{\pm} - \sum_{u \in \operatorname{Int}^{\mp} \gamma} (h_u^{\pm} - h_u^{\mp}) \right] \qquad \text{for} \quad \gamma \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \tag{19}$$

we may rewrite the partition function $\tilde{Z}_V^{\pm}$ of small contours in a simpler form as

$$\tilde{Z}_V^{\pm} = \sum_{\text{compatible } \tilde{\partial}} \prod_{\gamma \in \tilde{\partial}} \tilde{\rho}(\gamma) \tag{20}$$

There is a standard trick to get rid of the compatibility condition in (20), which is the basis of the Pirogov–Sinai approach.[9] Namely, we modify the activities further, defining

$$z(\gamma) = \tilde{\rho}(\gamma)\, \tilde{Z}^{\pm}_{\operatorname{Int}^{\mp} \gamma} \qquad \text{for} \quad \gamma \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \tag{21}$$

Then

$$\tilde{Z}_V^{\pm} = \sum_{\substack{\tilde{\partial} = \{\gamma\} \\ \gamma \text{ disjoint } \left\{ \substack{\text{positive} \\ \text{negative}} \right\} \text{ in } V}} \prod_{\gamma \in \tilde{\partial}} z(\gamma) \tag{22}$$

Thus, $\tilde{Z}_V^{\pm}$ becomes the standard polymer partition function.

We shall need estimates on the polymer activities $z(\gamma)$. First notice that for small contours $\gamma$, $|\operatorname{Int} \gamma| \leqslant O(L)\, |\gamma|$, and consequently

$$\sum_{u \in \operatorname{Int} \gamma} |\operatorname{Re}(h_u^{\pm} - h_u^{\mp})| \leqslant 4D\, |\operatorname{Int} \gamma| \leqslant O(L)\, D\, |\gamma| \leqslant \tfrac{1}{4} \eta \beta_0\, |\gamma| \tag{23}$$

if $\beta_0 > \bar{\beta}_0(\eta, D)$ and $L \leqslant \eta\beta_0 / O(1)\, D$, so that [see (14), (15)]

$$|\tilde{\rho}(\gamma)| \leqslant e^{-3\eta\beta_0 |\gamma|/4} \tag{24}$$

uniformly in $\Lambda$. Using this estimate, we shall show that

$$|z(\gamma)| \leqslant \exp(-\tfrac{2}{3} \eta \beta_0\, |\gamma|) \tag{25}$$

uniformly in $\Lambda$, so that the polymer (=contour) activities in (22) are strongly suppressed. Following Ref. 4, we shall establish (25) by induction in $|V(\gamma)|$. Supposing (25) to be valid for every $\gamma$ such that $|V(\gamma)| \leqslant k$, we may use the usual manipulations similar to those leading to (12) in order to exponentiate (22) (see, e.g., Ref. 10 or Ref. 5) for any $V$ with $|V| \leqslant k$:

$$\tilde{Z}_V^{\pm} = \exp\left[\sum_{C \subset V} \tilde{\phi}^{\pm}(C)\right] \tag{26}$$

The customary expression for $\tilde{\phi}^{\pm}(C)$ is

$$\tilde{\phi}^{\pm}(C) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\gamma_1,\ldots,\gamma_n) \\ \gamma_i \left\{\substack{\text{positive} \\ \text{negative}}\right\} \\ \cup \gamma_i = C}} a(\{\gamma_1,\ldots,\gamma_n\}) \prod_{i=1}^{n} z(\gamma_i) \tag{27}$$

with $a(\cdot)$ given by (13). The inequality (25) results in an estimate on the terms $\tilde{\phi}^{\pm}(C)$ contributing to (26):

$$|\tilde{\phi}(C)| \leqslant e^{-\eta\beta_0|C|/2} \tag{28}$$

uniformly in $\Lambda$. Let us only notice that both $\tilde{Z}_V^{\pm}$ with $|V| \leqslant k$ as well as $\tilde{\phi}^{\pm}(C)$ with $C \subset V$ depend only on contours $\gamma$ for which $|V(\gamma)| \leqslant k$, and thus, to prove (26) and (28), one may rely on (25) valid by the inductive hypothesis. From (26) and (28) it follows that

$$|\tilde{Z}_V^{\pm}| \leqslant \exp\left[\sum_{u \in V} \sum_{u \in C \subset V} |\tilde{\phi}^{\pm}(C)/|C|\right] \leqslant \exp[|V| \exp(-\tfrac{1}{3}\eta\beta_0)] \tag{29}$$

for $|V| \leqslant k$ and similarly

$$|\tilde{Z}_V^{\pm}| \geqslant \exp[-|V| \exp(-\tfrac{1}{3}\eta\beta_0)] \tag{30}$$

For a small contour $\gamma$ with $|V(\gamma)| \leqslant k+1$ one necessarily has $|\mathrm{Int}\,\gamma| \leqslant k$, and thus, in virtue of (21), (24), (29), and (30),

$$|z(\gamma)| \leqslant \exp[-\tfrac{3}{4}\eta\beta_0\,|\gamma| + 2\,|\mathrm{Int}\,\gamma| \exp(-\tfrac{1}{3}\eta\beta_0)] \tag{31}$$

As for small contours $|\mathrm{Int}\,\gamma| \leqslant O(L)\,|\gamma|$, the estimate (25) follows. The "uniformity" in $\Lambda$ of (25) follows the same way.

Let us rewrite (26) somewhat, introducing $\tilde{\phi}_V^{\pm}(C) \equiv \tilde{\phi}^{\pm}(C)\,|V \cap C|/|C|$ for $C, V \subset \Lambda$, $C \cap V \neq \varnothing$:

$$\begin{aligned}\tilde{Z}_V^{\pm} &= \exp\left[\sum_{C \subset \Lambda} \tilde{\phi}_V^{\pm}(C)\right] \exp\left[-\sum_{\Lambda \supset C \not\subset V} \tilde{\phi}_V^{\pm}(C)\right] \\ &= \exp\left(\sum_{u \in V} S_u^{\pm}\right) \sum_{\substack{\{C_\alpha\} \\ \Lambda \supset C_\alpha \not\subset V}} \prod_{\alpha} \{\exp[-\tilde{\phi}_V^{\pm}(C_\alpha)] - 1\}\end{aligned} \tag{32}$$

where

$$S_u^{\pm} = \sum_{\substack{C \ni u \\ C \subset \Lambda}} \tilde{\Phi}^{\pm}(C)/|C| \tag{33}$$

The inequality (28) implies that

$$|S_u^{\pm}| \leqslant e^{-\eta\beta_0/3} \tag{34}$$

uniformly in $\Lambda$. In the $\Lambda \nearrow \mathbb{R}^d$ limit

$$S_u^{\pm} \to S^{\pm} = \sum_{C \ni u} \tilde{\Phi}^{\pm}(C)/|C| \upharpoonright_{\Lambda = \mathbb{R}^d} \tag{35}$$

We may interpret $S^{\pm}$ as the pressure of the gas of small contours. Notice that $S^{+} + h^{+} \neq S^{-} + h^{-}$ in general: the pressure depends on the boundary condition because large contours close to $\partial V$ flipping the boundary conditions are not allowed.

## 5. COARSE GRAINING OF THE CONTOURS

Consider the numerator $Z_\Lambda^{\pm}(\{x_j\})$ of (6) (the denominator $Z_\Lambda^{\pm}$ is the same expression with $\{x_j\} = \varnothing$). Given $\partial = \{\gamma\}$, split it into $\bar{\partial} \cup \tilde{\partial}$, where $\bar{\partial}$ is the set of all big contours [with $|V(\gamma)| \geqslant L^d$; notice that the contours containing points $\{x_j\}$ are big by construction]. $\tilde{\partial}$ is composed of small contours $\gamma$ and if $\gamma \subset V^{\pm}(\bar{\partial})$, then also $V(\gamma) \subset V(\bar{\partial})$ (small contours cannot surround big ones). Resumming over $\tilde{\partial}$, we obtain

$$Z_\Lambda^{\pm}(\{x_j\}) = \sum_{\text{compatible } \bar{\partial}} \exp\left(\sum_{s=\pm} \sum_{u \in V^s(\bar{\partial})} h_u^s\right) \tilde{Z}_{V^+(\bar{\partial})}^{+} \tilde{Z}_{V^-(\bar{\partial})}^{-}$$
$$\times \prod_{\gamma \in \bar{\partial}} \rho_{\{x_j\} \cap \gamma}(\gamma) \tag{36}$$

Expressing the small-contour partition functions in (36) according to (32), we obtain

$$Z_\Lambda^{\pm}(\{x_j\}) = \sum_{\text{comp. } \bar{\partial}} \exp\left[\sum_s \sum_{u \in V^s(\bar{\partial})} (h_u^s + S_u^s)\right] \prod_{\gamma \in \bar{\partial}} \rho_{\{x_j\} \cap \gamma}(\gamma)$$
$$\times \sum_{\{C_\alpha\}, C_\alpha \not\subset V^+(\bar{\partial})} \prod_\alpha \{\exp[-\tilde{\Phi}_{V^+(\bar{\partial})}^{+}(C_\alpha)] - 1\}$$
$$\times \sum_{\{C_\beta\}, C_\beta \not\subset V^-(\bar{\partial})} \prod_\beta \{\exp[-\tilde{\Phi}_{V^-(\bar{\partial})}^{-}(C_\beta)] - 1\} \tag{37}$$

Consider the set

$$\left(\bigcup_{\gamma\in\bar{\partial}}\gamma\right)\cup\left(\bigcup_{\alpha}C_\alpha\right)\cup\left(\bigcup_{\beta}C_\beta\right)$$

and its connected components $\Gamma_\delta$. Thus, the $\Gamma_\delta$ are obtained by decorating big contours $\gamma\in\bar{\partial}$ by clusters $C_\alpha$ and $C_\beta$. The components of $\partial\Gamma_\delta$ inherit signs from the configuration $\bar{\partial}$, so that $\Gamma_\delta$ form naturally signed contours and $\Delta=\{\Gamma_\delta\}$ a compatible family of signed contours. Let us define for $\{y_k\}_{k=1}^K\subset\Gamma$ the activity of $\Gamma$ as

$$\begin{aligned}R_{\{y_k\}}(\Gamma)=&\sum_{\bar{\partial}}\sum_{\{C_\alpha\}}\sum_{\{C_\beta\}}\exp\left[\sum_{s}\sum_{u\in V^s(\bar{\partial})\cap\Gamma}(h_u^s+S_u^s)\right]\\&\times\prod_{\gamma\in\bar{\partial}}\rho_{\{y_k\}\cap\gamma}(\gamma)\prod_{\alpha}\{\exp[-\tilde{\Phi}^+_{V^+(\bar{\partial})}(C_\alpha)]-1\}\\&\times\prod_{\beta}\{\exp[-\tilde{\Phi}^-_{V^-(\bar{\partial})}(C_\beta)]-1\}\end{aligned}\tag{38}$$

where $\bar{\partial}$ runs through the compatible families of big contours (compatible also with the signs on $\partial\Gamma$) such that $\bigcup_{\gamma\in\bar{\partial}}\gamma\supset\{y_k\}$, $C_\alpha\cap V^+(\bar{\partial})\neq\varnothing$, $C_\alpha\not\subset V^+(\bar{\partial})$, $C_\beta\cap V^-(\bar{\partial})\neq\varnothing$, $C_\beta\not\subset V^-(\bar{\partial})$, and

$$\left(\bigcup_{\gamma\in\bar{\partial}}\gamma\right)\cup\left(\bigcup_{\alpha}C_\alpha\right)\cup\left(\bigcup_{\beta}C_\beta\right)=\Gamma$$

Let us estimate $R_{\{y_k\}}(\Gamma)$. We have

$$\begin{aligned}&\left|R_{\{y_k\}}(\Gamma)\exp\left[-\sum_{u\in\Gamma}(h_u^\pm+S_u^\pm)\right]\right|\\&\quad\leqslant\sum_{\bar{\partial},\{C_\alpha\},\{C_\beta\}}\left|\exp\left[\sum_{u\in V^\mp(\bar{\partial})\cap\Gamma}(h_u^\mp+S_u^\mp-h_u^\pm-S_u^\pm)\right]\right|\\&\quad\times\prod_{\gamma\in\bar{\partial}}\left\{\rho_{\{y_k\}\cap\gamma}(\gamma)\exp\left[-\sum_{u\in\gamma}(h_u^\pm+S_u^\pm)\right]\right\}\\&\quad\times\prod_{\alpha}[2\exp(-\tfrac{1}{2}\eta\beta_0|C_\alpha|)]\prod_{\beta}[2\exp(-\tfrac{1}{2}\eta\beta_0|C_\beta|)]\\&\quad\text{for }\begin{Bmatrix}\text{positive}\\\text{negative}\end{Bmatrix}\Gamma\end{aligned}\tag{39}$$

In the estimation of the right-hand side of (39), the sum over $\bar{\partial}$ may be replaced by

$$O(1)^{|\Gamma\setminus E(\{y_k\})|}$$

since only the part of $\bigcup_{\gamma \in \delta} \gamma$ outside $E(\{y_k\})$ as well as the signs in $\Gamma \backslash (\bigcup_{\gamma \in \bar{\delta}} \gamma)$ have to be chosen to determine $\bar{\delta}$. Similarly, the sum over $\{C_\alpha\}$ and $\{C_\beta\}$ may be replaced by the coefficient

$$2^{|V(\bar{\delta}) \cap \Gamma|} \prod_\alpha O(1)^{|C_\alpha|} \prod_\beta O(1)^{|C_\beta|}$$

since choosing a point of $V(\bar{\delta}) \cap \Gamma$ inside each $C_\alpha$ and $C_\beta$, we may estimate

$$\begin{aligned} &\sum_{\{C_\alpha\}} \sum_{\{C_\beta\}} 2^{-|V(\bar{\delta}) \cap \Gamma|} \prod_\alpha O(1)^{-|C_\alpha|} \prod_\beta O(1)^{-|C_\beta|} \\ &\quad \leqslant \sum_{X \subset V(\bar{\delta}) \cap \Gamma} 2^{-|V(\bar{\delta}) \cap \Gamma|} \\ &\quad\quad \times \prod_{x \in X} \left\{ \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{C \ni x} [2O(1)^{-|C|}]^n \right\} \leqslant 1 \end{aligned} \tag{40}$$

The first factor on the right-hand side of (39) is bounded by

$$\exp[(4D + 2e^{-\eta\beta_0/3}) \, |V^{\mp}(\bar{\delta}) \cap \Gamma|] \leqslant \prod_\alpha e^{5D|C_\alpha|} \prod_\beta e^{5D|C_\beta|}$$

Moreover, by (14) and (34),

$$\begin{aligned} &\left| \rho_{\{y_k\} \cap \gamma}(\gamma) \exp\left[ -\sum_{u \in \gamma} (h_u^{\pm} + S_u^{\pm}) \right] \right| \\ &\quad \leqslant \{A \exp[(3L)^d \exp(-\tfrac{1}{3}\eta\beta_0)]\}^{|\{y_k\} \cap \gamma|} \\ &\quad\quad \times \exp[-\tfrac{1}{2}\eta\beta_0 \, |\gamma \backslash E(\{y_k\})|] \exp[4D \, |E(\{y_k\}) \cap \gamma|] \end{aligned} \tag{41}$$

The last factor in (41) appears only for $\gamma$ of opposite exterior sign to that of $\Gamma$. It can be bounded by $\exp[O(L^d D) \, |\{y_k\} \cap \gamma|]$, leading to a drastic increase of $A$. If, however, all $\{y_k\}$ are in a single (or in, say, two) $L$-blocks so that the excluded volume is small, then, for $L \leqslant \eta\beta_0 / O(1) \, D$,

$$\sum_{\substack{\gamma \in \bar{\delta} \\ \gamma \left\{ \substack{\text{negative} \\ \text{positive}} \right\}}} \exp[4D \, |E(\{y_k\}) \cap \gamma|] \leqslant \exp\left[ \sum_{\gamma \in \bar{\delta}} \tfrac{1}{12}\eta\beta_0 \, |\gamma \backslash E(\{y_k\})| \right] \tag{42}$$

since each $\left\{ \substack{\text{negative} \\ \text{positive}} \right\}$ $\gamma$ has to be surrounded by a $\left\{ \substack{\text{positive} \\ \text{negative}} \right\}$ one occurring in $\bar{\delta}$.

Gathering all factors together and estimating similarly the difference between the finite- and infinite-volume expressions, we obtain the bound

$$\left| R_{\{y_k\}}(\Gamma) \exp\left[ -\sum_u (h_u^\pm + S_u^\pm) \right] \right| \leqslant A'^{|\{y_k\}|} \exp[-\tfrac{1}{3}\eta\beta_0 |\Gamma \backslash E(\{y_k\})|] \qquad \text{for} \quad \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \Gamma \tag{43}$$

uniformly in $\Lambda$, with

$$A' = A \exp[(3L)^d \exp(-\tfrac{1}{3}\eta\beta_0)] \exp[O(L^d D)] \tag{44}$$

with the last factor absent if all $\{y_k\}$ are in one or two $L$-blocks.

It is easy to see that (37) may be rewritten as

$$Z_\Lambda^\pm(\{x_j\}) = \sum_{\substack{\text{compatible } \Delta \\ \bigcup_{\Gamma \in \Delta} \Gamma \supset \{x_j\}}} \exp\left[ \sum_{s = \pm} \sum_{u \in V^s(\Delta)} (h_u^s + S_u^s) \right] \prod_{\Gamma \in \Delta} R_{\{x_j\} \cap \Gamma}(\Gamma) \tag{45}$$

so that the resummation of small contours gives for the correlation functions an expression analogous to (6), but with big contours only. The point is that big contours represent nonlocal contributions to the interaction (the local ones are contained in the external field). In the renormalization group language they correspond to irrelevant operators which contract under *coarse graining.*

We shall use the most straightforward coarse graining procedure possible. Given a compatible family of contours $\Delta$ in (45), consider the union $G$ of the $\mathscr{L}$-blocks, $\mathscr{L} = l^m \leqslant L$, which intersect $\bigcup_{\Gamma \in \Delta} \Gamma$ and of the $L'\mathscr{L}$-blocks containing the points $\{x_j\}$ or intersecting the latter ones. The condition $L' \geqslant L$ will serve to define small and big contours on the next scale, and we shall choose it later. The connected components of $G$ have the form $\mathscr{L}\gamma'$, where $\gamma'$ is a connected set $\subset \Lambda' \equiv \mathscr{L}^{-1}\Lambda$ built of the closed unit lattice cubes and $L'$-blocks containing the points $x_j' \equiv \mathscr{L}^{-1} x_j$. The components of $\partial G$ and hence also those of $\partial\gamma'$ inherit signs from those of the family $\Delta$. The $\partial' = \{\gamma'\}$ becomes a compatible family of contours in $\Lambda'$. Define the coarse-grained activities by

$$\rho'_{\{x_j'\} \cap \gamma'}(\gamma') = \sum_\Delta \exp\left[ \sum_s \sum_{u \in V^s(\Delta) \cap \mathscr{L}\gamma'} (h_u^s + S_u^s) \right] \prod_{\Gamma \in \Delta} R_{\{x_j\} \cap \Gamma}(\Gamma) \tag{46}$$

where we sum over the collections $\Delta$ of signed contours $\Gamma$ in $\mathscr{L}\gamma'$ compatible mutually and with the signs on $\mathscr{L}\partial\gamma'$. The further constraint is that $\mathscr{L}\gamma'$ has to be the union of the $\mathscr{L}$-blocks intersecting $\bigcup_{\Gamma \in \Delta} \Gamma$ not disjoint with $\mathscr{L}\gamma'$ and of the $L'\mathscr{L}$-blocks in $\Lambda$ containing the $x_j$ or intersecting the latter ones not disjoint with $\mathscr{L}\gamma'$. Let also for $u \in \Lambda'$

$$h_u^{\pm\prime} = \sum_{\substack{y \text{ in the } \mathscr{L}\text{-block} \\ \text{centered at } \mathscr{L}u}} (h_y^{\pm} + S_y^{\pm}) \tag{47}$$

We may rewrite (45) and consequently $\langle \prod_j \sigma_{x_j} \rangle_\Lambda^\pm$ on the next scale as

$$\left\langle \prod_j \sigma_{x_j} \right\rangle_\Lambda^\pm = \left\{ \sum_{\substack{\text{compatible } \partial' \\ \bigcup_{\gamma' \in \partial'} \gamma' \supset \{x_j'\}}} \exp\left( \sum_{s = \pm} \sum_{y \in V^s(\partial')} h_u^{s'} \right) \prod_{\gamma' \in \partial'} \rho'_{\{x_j'\} \cap \gamma'}(\gamma') \right\}$$
$$\times \left\{ \sum_{\text{compatible } \partial'} \exp\left( \sum_s \sum_{u \in V^s(\partial')} h_u^{s'} \right) \prod_{\gamma' \in \partial'} \rho'(\gamma') \right\}^{-1} \tag{48}$$

Let us see what we have gained by application of the coarse-graining procedure.

We shall start by estimation of $\rho'_\varnothing(\gamma') \equiv \rho'(\gamma')$ as given by (46). In that case, it is easy to see one has

$$|\gamma'| < O(\mathscr{L}^{-1}) \sum_{\Gamma \in \Delta} |\Gamma| \tag{49}$$

since all contours $\Gamma$ are big [(49) obviously fails for small contours]. The external field contributions to

$$\rho'(\gamma') \exp\left( -\sum_{u \in \gamma'} h_u^{\pm\prime} \right) \qquad \text{with} \quad \gamma' \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix}$$

can be easily estimated by $\exp(-\varepsilon\beta_0 \sum_{\Gamma \in \Delta} |\Gamma|)$, since $\mathscr{L} \leqslant \eta\beta_0/O(1)\, D$. The sum over the collections $\Delta$ is controlled in a straightforward way:

$$\sum_\Delta \prod_{\Gamma \in \Delta} \exp(-\varepsilon\beta_0 |\Gamma|) \leqslant \exp\{O[\exp(-\varepsilon\beta_0)]\, |\mathscr{L}\gamma'|\} \tag{50}$$

Using also (49), we infer that

$$\left| \rho'(\gamma') \exp\left( -\sum_{u \in \gamma'} h_u^{\pm\prime} \right) \right| \leqslant \exp[-\tfrac{1}{4}\eta\mathscr{L}\beta_0 |\gamma'|/O(1)] = \exp(-\beta_0' |\gamma'|) \tag{51}$$

with

$$\beta_0' = \mathscr{L}\beta_0/O(1) \tag{52}$$

The case of $\rho'_{\{y_k'\}}(\gamma')$ is somewhat more complicated because of the presence of the excluded volumes. However, it is easy to see that every

$\mathscr{L}$-block of $\mathscr{L}(\gamma'\backslash E(\{y'_k\}))$ has to intersect a big $\Gamma \in \Delta$, $\Gamma \cap \{y_k\} = \varnothing$, or a $\Gamma \in \Delta$, $\Gamma \cap \{y_k\} \neq \varnothing$, such that $\Gamma\backslash E(\{y_k\})$ is sufficiently big. Thus,

$$|\gamma'\backslash E(\{y'_k\})| < O(\mathscr{L}^{-1}) \left[ \sum_{\Gamma \in \Delta} |\Gamma\backslash E(\{y_k\})| \right] \tag{53}$$

in this case. The sum over configurations $\Delta = \{\Gamma\}$ in (46) can be easily estimated as in (50) by

$$\exp\{-O[\exp(-\varepsilon\beta_0)]\ |\mathscr{L}\gamma'\backslash E(\{y_k\})|\}$$

The external field contributions in $\mathscr{L}(\gamma'\backslash E(\{y'_k\}))$ are absorbed using again

$$\prod_{\Gamma \in \Delta} \exp[-\varepsilon\beta_0\ |\Gamma\backslash E(\{y_k\})|]$$

The only problem arises from the possible external field contributions from $\mathscr{L}E(\{y'_k\})$, which may be absorbed into a significant increase of the constant $A$. If, however, all $\{y_k\}$ are in one (or two) $L$-blocks, then these contributions can be again absorbed as in (42), provided that

$$L'\mathscr{L} \leqslant \eta\beta_0/O(1)\, D \tag{54}$$

with $O(1)$ big enough. This is the most stringent restriction on the scales $\mathscr{L}$, $L$, $L'$, $O(1) \ll \mathscr{L} \leqslant L \leqslant L'$, which we shall encounter. Summarizing,

$$\left|\rho'_{\{y'_k\}}(\gamma') \exp\left(-\sum_{u \in \gamma'} h_u^{\pm\prime}\right)\right| \leqslant A'^{|\{y'_k\}|} \exp[-\eta\beta'_0\ |\gamma'\backslash E(\{y'_k\})|] \qquad \text{for} \quad \gamma' \begin{Bmatrix}\text{positive}\\ \text{negative}\end{Bmatrix} \tag{55}$$

uniformly in $\Lambda$ with $A' = \{\exp O[(L'\mathscr{L})^d D]\} A$ or $A' = \{\exp O[\exp(-\varepsilon\beta_0)]\} A$ if all $y_k$ are in one or two $L$-blocks.

There are two crucial effects of the coarse graining. The first is a decrease in the effective temperature:

$$\beta_0 \to \beta'_0 = \mathscr{L}\beta_0/O(1) \tag{56}$$

The other important effect is a change, according to (47), of the infinite-volume field

$$h^\pm \to h^{\pm\prime} = \mathscr{L}^d[h^\pm + O(e^{-\eta\beta_0/3})] \tag{57}$$

Note also that in finite volume,

$$|h_u^{\pm\prime} - h^{\pm\prime}| \leqslant \exp[-\varepsilon\beta_0' \operatorname{dist}(u, \partial\Lambda')] \tag{58}$$

if $u$ is not a site adjacent to $\partial\Lambda'$. Thus, $h = \frac{1}{2}(h^+ - h^-)$ is a *relevant variable*, which expands under the coarse graining. Its control will proceed by a method used by Bleher and Sinai.[1]

## 6. THE INDUCTIVE PROCEDURE

Given initial $h$ with $|h_0| < 2D$, we shall distinguish three possible outcomes of the coarse graining described in the previous section.

(i) $h_0' \equiv \operatorname{Re} h' \equiv \frac{1}{2}\operatorname{Re}(h^{+\prime} - h^{-\prime}) \geqslant D$. Then the expansion of Section 3 establishes the existence of the $(+)$ phase.

(ii) $-h_0' \geqslant D$. Then, by the same argument, the $(-)$ phase exists.

(iii) $|h_0'| < 2D$. This means that

$$|h_0| < 2\mathscr{L}^{-d}D + O(e^{-\eta\beta_0/3}) < D$$

Since our estimates worked uniformly for $|h_0| < 2D$ and all entries were analytic in $h$, we easily infer (using the Cauchy estimate) that for $|h_0| < D$

$$dh'/dh = \mathscr{L}^d + O(e^{-\varepsilon\beta_0}) \tag{59}$$

It follows easily that the inverse function $h(h')$ is uniquely determined for $|h_0'| < 2D$, and consequently all quantities can be expressed as analytic functions of $h'$ there. Fix $\operatorname{Im} h$ (i.e., $\operatorname{Im} J_{\{x\}}$ in the original spin model). For $h_0$ running through $(-D, D) \equiv I$, $h'$ sweeps an analytic curve $\mathscr{M}_1$ which forms an angle $\leqslant O(e^{-\varepsilon\beta_0})$ with the real axis. Consider the function

$$I \ni h_0 \xrightarrow{f_1} h_0' \in \mathbb{R}^1 \tag{60}$$

Clearly

$$df_1/dh_0 = \mathscr{L}^d + O(e^{-\varepsilon\beta_0}) \tag{61}$$

so that $f_1$ expands and is increasing. Let $I_0^0 = \bar{I}$ and $I_1^0 = f_1^{-1}(\bar{I}) \subset I \subset I_0^0$. To the right of $I_1^0$ in $I_0^0$, $h_0' \geqslant D$ and we have the $(+)$ phase; to the left of $I_1^0$, we have the $(-)$ phase. The case of $h_0 \in I_1^0$ will be decided in the next steps.

For $|h_0'| < 2D$, we shall apply another coarse-graining step. The choice of the scales $\mathscr{L}'$, $L''$ characterizing this step, $O(1) \leqslant \mathscr{L}' \leqslant L' \leqslant L''$, has to satisfy

$$L''\mathscr{L}' \leqslant \eta\beta_0'/O(1)\, D \tag{62}$$

[see (54)]. For the sake of the present section, we could choose all those scales equal. However, (62) allows a superexponential growth in subsequent scales. For example, $\mathscr{L} = L^{3/4} < L < L' = L^{5/4}$ leads to $L''\mathscr{L}' = L'^{5/4}L'^{3/4} = L'\mathscr{L}L^{1/2}$, so that (54) will imply (62). We shall use the possibility of choosing rapidly growing scales in the analysis of the decay of the correlation functions in the next section.

The second coarse-graining step differs from the first one only in one respect, namely that the external field $h_u^{\pm\prime}$ in a finite volume $\Lambda'$ might depend on $u$ everywhere in $\Lambda'$. However, due to (58), the differences

$$|\mathrm{Re}(h_u^{+\prime} - h_u^{-\prime})| \leqslant 2D + 2\exp[-\varepsilon\beta_0' \,\mathrm{dist}(u, \partial\Lambda')] \tag{63}$$

are well under control for $u$ not adjacent to $\partial\Lambda'$. Since we shall have to estimate $|\mathrm{Re}(h_u^{+\prime} - h_u^{-\prime})|$ only in the interiors of contours flipping signs, (63) will be sufficient. For the sites not adjacent to the boundary of the volume, (58) clearly iterates:

$$\begin{aligned} |h_u^{\pm\prime\prime} - h^{\pm\prime\prime}| &\leqslant \sum_{\substack{y \text{ in the } \mathscr{L}' \text{ block} \\ \text{centered at } \mathscr{L}'u}} |(h_y^{\pm\prime} - h^{\pm\prime}) + (S_y^{\pm\prime} - S^{\pm\prime})| \\ &\leqslant \sum_y \{1 + O[\exp(-\tfrac{1}{3}\eta\beta_0')]\} \exp[-\varepsilon\beta_0' \,\mathrm{dist}(u, \partial\Lambda')] \\ &\leqslant \exp[-\varepsilon\beta_0'' \,\mathrm{dist}(u, \partial\Lambda'')] \end{aligned} \tag{64}$$

because

$$\mathrm{dist}(y, \partial\Lambda') > \frac{\mathscr{L}'}{O(1)} \mathrm{dist}(u, \partial\Lambda''), \qquad \mathrm{dist}(u, \partial\Lambda'') \geqslant \tfrac{3}{2}$$

Again

$$dh''/dh' = \mathscr{L}'^d + O[\exp(-\varepsilon\beta_0')] \tag{65}$$

if $|h_0'| < D$, and for $|h_0''| < 2D$ all quantities are analytic functions of $h_0''$. The piece of the curve $\mathscr{M}_1$ satisfying $|h_0'| < D$ is clearly mapped into an analytic curve $\mathscr{M}_2$ of $h''$, which forms an angle

$$\leqslant O[\exp(-\varepsilon\beta_0)] + O[\exp(-\varepsilon\beta_0')]$$

with the real axis. Consider the function $f_2$,

$$\overset{\curvearrowleft\; f_2 \;\curvearrowright}{I \ni h_0' \mapsto \underset{\mathscr{M}_1}{\underset{\cap}{h'}} \mapsto \underset{\mathscr{M}_2}{\underset{\cap}{h''}} \mapsto h_0'' \in \mathbb{R}^1} \tag{66}$$

Clearly, by the chain rule and (65),

$$\frac{df_2}{dh'_0} = \mathscr{L}'^d + O[\exp(-\varepsilon\beta_0)] + O[\exp(-\varepsilon\beta'_0)] \tag{67}$$

Let $I_0^1 = \bar{I}$ and $I_1^1 = f_2^{-1}(\bar{I}) \subset I \subset I_0^1$. To the right (left) of $I_1^1$ in $I_0^1$, $\mathrm{Re}\, h''_0 \geqslant D$ $(-\mathrm{Re}\, h''_0 \geqslant D)$ and the $+$ $(-)$ phase exists. In $I_1^1$ we have to coarse grain again.

Let $I_2^0 = f_1^{-1}(I_1^1)$, $I_0^0 \supset I_1^0 \supset I_2^0$. Proceeding further, we construct a sequence of functions

$$f_n: I \to \mathbb{R}^1 \tag{68}$$

such that

$$\frac{df_{n+1}}{dh_0^{(n)}} = (\mathscr{L}^{(n)})^d + \sum_{k=0}^{n} O[\exp(-\varepsilon\beta_0^{(k)})] \tag{69}$$

Since

$$\sum_{k=0}^{\infty} O[\exp(-\varepsilon\beta_0^{(k)})] = O[\exp(-\varepsilon\beta_0)]$$

all $f_n$ expand and are increasing. Defining $I_{q+1}^{p-1} = f_p^{-1}(I_q^p)$, we obtain a sequence of closed, nonempty intervals

$$I_0^0 \supset I_1^0 \supset I_2^0 \supset \cdots \supset I_n^0 \supset \cdots$$

with rapidly decreasing length. Define (for each $\mathrm{Im}\, h$) the critical value $h_{0c}$ by

$$\{h_{0c}\} = \bigcap_n I_n^0$$

For $h_0 > h_{0c}$, at some stage, we shall be able to apply the single-scale, large-external-field expansion of Section 3 to the correlation functions with $(+)$ boundary condition and establish existence of the $(+)$ phase. Similarly, for $h_0 < h_{0c}$, the $(-)$ phase exists.

The interesting question is the dependence of $h_{0c}$ on $\mathrm{Im}\, h$. The phase separation curve $h_{0c}(\mathrm{Im}\, h)$ can be obtained as the limit when $n \to \infty$ of the subsequent inverse images of the line $\{h^{(n)}: h_0^{(n)} = 0\}$ under the mapping $h \to h' \to \cdots \to h^{(n)}$. It is again easy to see that these curves from an angle bounded by $\sum_{k=0}^{\infty} O[\exp(-\varepsilon\beta_0^{(k)})]$ with the imaginary axis and that they converge uniformly with first (and in fact all) derivatives to the curve $h_{0c}(\mathrm{Im}\, h)$. We conclude that the phase separation curve is $(C^\infty-)$ smooth.

On this curve, i.e., for $h_0 = h_{0c}$, we may continue the coarse graining forever, or, in the finite volume, $n$ times, until the volume shrinks to an $O(L^{(n)})$ block centered at the origin. Consider the volumes big enough so that by that time all $x_j^{(n)}$, $j = 1,..., J$, are in the $L^{(n)}$-block centered at the origin. Let $\gamma_0^{(n)}$ be the contour composed of the $3L^{(n)}$-block centered at zero with $\pm$ sign on $\partial\gamma_0^{(n)}$. Then

$$\left\langle \prod_j \sigma_{x_j} \right\rangle_\Lambda^\pm = \left\{ \exp\left[ -\sum_{u \in \gamma_0} h_u^{\pm(n)} \right] \rho^{(n)}_{\{x_j^{(n)}\}}(\gamma_0^{(n)}) + B_{\{x_j^{(n)}\}} \right\} \Big/ (1 + B) \qquad (70)$$

where $B_{\{x_j^{(n)}\}}$ resums the contributions of all configurations different from the minimal one consisting of $\gamma_0^{(n)}$ only. Similarly, $B$ resums all the contributions of nonempty configurations of contours. Since for all contours inside the $O(L^{(n)})$-block the contour activities dominate the external field contributions, we easily see that

$$|B_{\{x_j^{(n)}\}}| \leqslant (A^{(n)})^{|\{x_j^{(n)}\}|}\, O[\exp(-\varepsilon\beta_0^{(n)})] \qquad (71)$$

with $A^{(n)}$ uniformly bounded and similarly

$$|B| \leqslant O[\exp(-\varepsilon\beta_0^{(n)})]$$

Moreover,

$$\begin{aligned} &\left| \left[ \exp\left( -\sum_{u \in \gamma_0} h_u^{\pm(n)} \right) \right] \rho^{(n)}_{\{x_j^{(n)}\}}(\gamma_0^{(n)}) \restriction_{\Lambda^{(n)}} \right. \\ &\qquad \left. - [\exp(-|\gamma_0^{(n)}|\, h^{\pm(n)})]\, \rho^{(n)}_{\{x^{(n)}\}}(\gamma_0^{(n)}) \restriction_{\mathbb{R}^d} \right| \\ &\leqslant (A^{(n)})^{|\{x_j^{(n)}\}|} \exp[-\varepsilon\beta_0^{(n)} d(\gamma_0^{(n)}, \partial\Lambda^{(n)})] \end{aligned} \qquad (72)$$

and

$$[\exp(-|\gamma_0^{(n)}|\, h^{\pm(n)})]\, \rho^{(n)}_{\{x_j^{(n)}\}}(\gamma_0^{(n)})$$

have a limit when $n \to \infty$; the main contribution to

$$[\exp(-|\gamma_0^{(n+1)}|\, h^{\pm(n+1)})]\, \rho^{(n+1)}_{\{x_j^{(n+1)}\}}(\gamma_0^{(n+1)})$$

comes from

$$[\exp(-|\gamma_0^{(n)}|\, h^{\pm(n)})]\, \rho^{(n)}_{\{x_j^{(n)}\}}(\gamma_0^{(n)}) \exp(-\mathscr{L}^{(n)d}\, |\gamma_0^{(n+1)}|\, S^{\pm(n)})$$

Thus, for $\operatorname{Re} h_0 = \operatorname{Re} h_{0c}$,

$$\lim_{\Lambda \nearrow \mathbb{R}^d} \left\langle \prod_j \sigma_{x_j} \right\rangle_\Lambda^\pm = \lim_{n \to \infty} [\exp(-|\gamma_0^{(n)}|\, h^{\pm(n)})]\, \rho^{(n)}_{\{x_j^{(n)}\}}(\gamma_0^{(n)}) \qquad (73)$$

This establishes the existence of the $(+)$ and the $(-)$ phases in this case. By looking at $\langle\sigma_0\rangle_\Lambda^\pm$ it is easy to see that $\langle\sigma_0\rangle^+ \neq \langle\sigma_0\rangle^-$, so that the phases are different.

Our iterative approach may be used to analyze various properties of the phases constructed above. An example of such an analysis is given in what follows.

## 7. FINITENESS OF THE CORRELATION LENGTH

It is easy to establish the exponential decay of the correlation functions of our model uniformly in $h_0$ around the first-order transition point. To this end, notice that we could have extracted an additional factor

$$\exp\{-2\eta\beta_0 \operatorname{diam}[\gamma\backslash E(\{y_k\})]\}$$

on the right-hand side of (7) [and of (14)], which would be inherited by the right-hand side of (11) (with $\gamma \to X_\alpha$). In the large-external-field expansion of Section 3, we obtain for the connected expectation [compare (12)]

$$\begin{aligned}
&\Big\langle \prod_{j_1} \sigma_{x_{j_1}}; \prod_{j_2} \sigma_{x_{j_2}} \Big\rangle^\pm_{\mathrm{conn}} \\
&\quad= \sum_{\substack{\{X_\alpha\}\ \text{disjoint} \\ \bigcup X_\alpha \supset \{x_{j_1}, x_{j_2}\} \\ X_\alpha \cap (\{x_{j_1}, x_{j_2}\}) \neq \varnothing \\ \text{some } X_\alpha \text{ intersect both } \{x_{j_1}\} \text{ and } \{x_{j_2}\}}} \prod_\alpha \rho^\pm_{\{x_{j_1}, x_{j_2}\} \cap X_\alpha}(X_\alpha) \\
&\qquad\times \sum_{n=0}^\infty \frac{1}{n!} \sum_{(Y_1, \dots, Y_n)} a\Big(\Big\{\bigcup X_\alpha, Y_m\Big\}\Big) \prod_{m=1}^n \rho^\pm(Y_m) \\
&\quad+ \sum_{\substack{\{X_{\alpha_1}\}\ \text{disjoint} \\ \bigcup X_{\alpha_1} \supset \{x_{j_1}\} \\ X_{\alpha_1} \cap \{x_{j_1}\} \neq \varnothing}} \sum_{\substack{\{X_{\alpha_2}\}\ \text{disjoint} \\ \bigcup X_{\alpha_2} \supset \{x_{j_2}\} \\ X_{\alpha_2} \cap \{x_{j_2}\} \neq \varnothing}} \prod_{i=1}^2 \prod_{\alpha_i} \rho^\pm_{\{x_{j_i}\} \cap X_{\alpha_i}}(X_{\alpha_i}) \\
&\qquad\times \sum_{n=0}^\infty \frac{1}{n!} \sum_{(Y_1, \dots, Y_n)} a\Big(\Big\{\bigcup X_{\alpha_1}, \bigcup X_{\alpha_2}, Y_m\Big\}\Big) \prod_{m=1}^n \rho^\pm(Y_m)
\end{aligned} \tag{74}$$

The connectedness structure in (74) allows us to extract the additional factor

$$\exp\{-2\eta\beta_0[d(\{x_{j_1}\}, \{x_{j_2}\}) - O(L)]\} \tag{75}$$

from the standard estimate of (74) uniform in the distance between $\{x_{j_1}\}$ and $\{x_{j_2}\}$ [$O(L)$ is due to the excluded volumes around $\{x_{j_1}, x_{j_2}\}$].

In order to establish the exponential decay of the correlations, we have to see how the extra factor $\exp\{-2\eta\beta_0 \operatorname{diam}[\gamma\backslash E(\{y_k\})]\}$ in (14), which immediately generates an extra $\exp\{-2\eta\beta_0 \operatorname{diam}[\Gamma\backslash E(\{y_k\})]\}$ in (43), iterates under the coarse graining. Let $\Delta$ be the family of contours $\Gamma$ that under blocking gives a new scale contour $\gamma'$ [see (46)]. It is easy to see that

$$\begin{aligned}\mathscr{L} \operatorname{diam}&[\gamma'\backslash E(\{x_j'\} \cap \gamma')]\\ &\leqslant \sum_{\Gamma \in \Delta} \{\operatorname{diam}[\Gamma\backslash E(\{x_j\} \cap \Gamma)] + O(\mathscr{L})\}\\ &\leqslant [1 + O(\mathscr{L}/L)] \sum_{\Gamma \in \Delta} \operatorname{diam}[\Gamma\backslash E(\{x_j\} \cap \Gamma)] \end{aligned} \tag{76}$$

since for every $\Gamma$ intersecting $\mathscr{L}(\gamma'\backslash E(\{x_j'\} \cap \gamma'))$ one has $\operatorname{diam}[\Gamma\backslash E(\{x_j\} \cap \Gamma)] \geqslant O(L)$. Hence, on the right-hand side of (55), we obtain an extra factor

$$\exp\left\{-\frac{2\eta\mathscr{L}\beta_0}{1 + O(\mathscr{L}/L)} \operatorname{diam}[\gamma'\backslash E(\{y_k'\})]\right\} \tag{77}$$

and after $n$ iterations, an extra factor

$$\exp\left\{-\frac{2\eta\mathscr{L}\cdots\mathscr{L}^{(n-1)}\beta_0}{[1 + O(\mathscr{L}/L)]\cdots[1 + O(\mathscr{L}^{(n-1)}/L^{(n-1)})]} \operatorname{diam}[\gamma\backslash E(\{y_k^{(n)}\})]\right\} \tag{78}$$

will appear on the right-hand side of (14). Since $\mathscr{L}^{(k)}/L^{(k)}$ decreases rapidly with our superexponential growth of the scales and

$$d(\{x_{j_1}^{(n)}\}, \{x_{j_2}^{(n)}\}) = (\mathscr{L}\cdots\mathscr{L}^{(n-1)})^{-1}\, d(\{x_{j_1}\}, \{x_{j_2}\}) \tag{79}$$

we obtain a factor

$$\exp\{-\eta\beta_0[d(\{x_{j_1}\}, \{x_{j_2}\}) - O(\mathscr{L}\cdots\mathscr{L}^{(n-1)}L^{(n)})]\}$$

when estimating

$$\left\langle \prod_{j_1} \sigma_{x_{j_1}}; \prod_{j_2} \sigma_{x_{j_2}} \right\rangle_{\text{conn}}^{\pm}$$

after $n$ coarse-graining steps by the large-external-field expansion of Section 3. Thus, the uniform exponential decay of correlation functions follows for $h_0 \neq h_{0c}$.

Establishing the exponential decay for $h_0 = h_{0c}$ requires somewhat different treatment of the correlation functions, without use of the excluded volumes (their introduction allowed a simplified treatment of small contours). We shall not pursue this point here.

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